LINEAR APPROXIMATION FOR THE FLOW OF A CONDUCTING GAS WITH A HIGH MAGNETIC REYNOLDS NUMBER THROUGH A CHANNEL

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In this paper, we discuss the flow of a nonviscous and non-heat-conducting gas through a channel of variable cross section under the influence of a transverse magnetic field.
For high magnetic Reynolds numbers, the flow is shown to consist of a core and current layers at the electrodes and at the fixed channel walls. The distributions of currents and other parameters in the core and in the current layers are found analytically, in a linear approximation. The Joule dissipation in the current layers may be more intense than that in the core. The longitudinal currents and Joule dissipation increase with increasing Hall parameter in the electrode layers.

Zhigulev [1] has shown that magnetic boundary layers may form in the flow of a conducting gas when there is a high magnetic Reynolds number ($R_m \gg 1$). He illustrated this situation by the shielding of a plasma flow from the magnetic fields produced near a plate which is electrically isolated from the plasma and through which a current is flowing. In an incompressible fluid, the layer thickness is proportional to $R_m^{-1/2}$. Morozov and Shubin [2] have offered a linear-approximation treatment of the structure of the electromagnetic near-electrode layers which arise during the flow of a nonviscous plasma with a high R_m and a small "exchange" parameter $\xi \approx H/R_m$, for flow transverse to a magnetic field and near a corrugated wall. They pointed out the possible formation of "dissipationless" near-electrode layers with thicknesses on the order of the Debye or electron Larmor radii, and a "dissipative" layer whose thickness increases along the length of the electrodes and is proportional to $(R_m c_B^2/c_T^2)^{-1/2}$, where c_B and c_T are the magnetic and thermal sound velocities. Morozov and Shubin studied the properties of dissipationless and dissipative electromagnetic layers at segmented accelerator electrodes through which a current is passing, for an arbitrary "exchange" parameter, in [2] and [3], respectively. The exchange parameter ξ was found in [4].

Such layers should also exist at solid electrodes and at the nonconducting walls of an accelerator channel. Study of the two-dimensional flow in a channel is significantly simplified when such layers are present.

1. Let us consider the planar flow of a conducting gas transverse to a magnetic field in a channel of variable cross section (Fig. 1). The channel walls may be either electrodes or insulators. We make the following assumptions: 1) The flow is quasi neutral and steady state. 2) The gas is nonviscous and non-heat-conducting. 3) The external magnetic field is uniform and directed along the z-axis, and there is no current along this axis, so the net magnetic field is along z and is a function of x and y. 4) The terms in the Ohm equation proportional to the gradients of the pressure and the electron temperature, and the terms related to the slipping of ions are negligible [5]. In dimensionless form, this equation is

$$\frac{1}{\sigma R_m}\nabla\times\mathbf{B}+\frac{H}{\alpha\rho R_m}(\nabla\times\mathbf{B})\times\mathbf{B}=-\nabla\varphi+\mathbf{v}\times\mathbf{B}. \quad (1.1)$$

5) The potential drop in the layers thinner than the dissipative layer is negligible.

Under these conditions, the flow is described by the following system of equations:

$$\frac{1}{\sigma R_m}\triangle\mathbf{B}-\frac{1}{R_m}\nabla\frac{1}{\sigma}\times(\nabla\times\mathbf{B})+$$
$$+\frac{H}{R_m}\nabla\frac{1}{\alpha\rho}\times\nabla\frac{B^2}{2}=(\mathbf{v}\nabla)\mathbf{B}+\mathbf{B}(\nabla\mathbf{v}),$$
$$\rho(\mathbf{v}\nabla)\mathbf{v}=-\frac{\nabla(\rho T)}{\gamma M^2}-A^2\nabla\frac{B^2}{2},\qquad \nabla(\rho\mathbf{v})=0,$$
$$\rho T(\mathbf{v}\nabla)\ln\frac{T}{\rho^{\gamma-1}}=\frac{A^2M^2}{R_m}\gamma(\gamma-1)\frac{(\nabla\times\mathbf{B})^2}{\sigma}. \quad (1.2)$$

All these quantities are dimensionless. The density ρ, the velocity v, the magnetic field **B**, the temperature T, and the degree of ionization α are expressed in units of their values in the initial cross section; these values are denoted by the subscript "0." The channel width at this cross section is used as the unit of length.

The similarity criteria are the following quantities: R_m, the magnetic Reynolds number; M, the Mach number; A, the Alvén number; H, the Hall parameter; and γ, the ratio of heat capacities:

$$R_m=\sigma_0 v_0 h\mu,\qquad M^2=\frac{v_0^2}{\gamma R T_0},$$
$$A^2=\frac{B_0^2}{\eta\rho_0 v_0^2},\qquad H=\frac{\sigma_0 m_i B_0}{\alpha\rho_0 e}.$$

If the changes in the dimensionless quantities in the channel are small in comparison with unity; that is, if

$$|v-1|\sim|\rho-1|\sim|B-1|\sim\varepsilon\leqslant 1,$$

then these equations can be linearized. In this procedure, the nonlinear term with the Hall effect disappears from the first, or induction equation, if the parameter H/R_m is not too large; that is, if $H/R_m \ll 1/\varepsilon$. Similarly, the nonlinear term describing Joule dissipation disappears from the energy equation when $M^2A^2/R_m \ll 1/\varepsilon$. After some transformations, the system of linearized equations becomes

$$\frac{1}{R_m}\triangle B=\frac{M^2}{M^2-1}\frac{\partial v}{\partial y}+\frac{M^2(1-A^2)-1}{M^2-1}\frac{\partial B}{\partial x},$$
$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{M^2-1}\frac{\partial^2 v}{\partial y^2}=-\frac{A^2M^2}{M^2-1}\frac{\partial^2 B}{\partial x\,\partial y},$$
$$\frac{\partial\rho}{\partial x}=\frac{A^2M^2}{M^2-1}\frac{\partial B}{\partial x}-\frac{M^2}{M^2-1}\frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial x}=-\frac{1}{M^2}\frac{\partial\rho}{\partial x}-A^2\frac{\partial B}{\partial x}. \quad (1.3)$$

Here u and v are the longitudinal and transverse components of the flow velocity (Fig. 1).

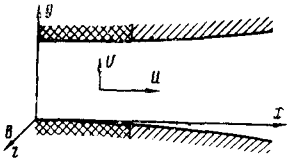

Fig. 1

The boundary conditions are as follows: 1) The velocity component normal to the channel wall (impermeable to the gas) is zero; in the linear approximation; this is described by

$$v = f'(x), \tag{1.4}$$

where $y = f(x)$ is the shape of the channel wall. 2) At a nonconducting wall, the normal component of the electric current is zero; in the linear approximation, this leads to the boundary condition

$$\partial B/\partial x = 0, \tag{1.5}$$

that is, at this wall the net magnetic field is constant:

$$B = \text{const.}$$

3) At an electrode, the tangential component of the electric field intensity is continuous; at an ideally conducting electrode, there is no electric field, and the boundary condition for the magnetic field in the linear approximation becomes

$$\frac{\partial B}{\partial y} = H\frac{\partial B}{\partial x}. \tag{1.6}$$

2. When $R_m \gg 1$, the induction equation simplifies, and system (1.3) reduces to the following, after some conversions:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{M_B{}^2 - 1}\frac{\partial^2 v}{\partial y^2} = 0, \qquad \frac{\partial B}{\partial x} = -\frac{M_B{}^2}{M_B{}^2 - 1}\frac{\partial v}{\partial y},$$
$$\rho = \rho(0, y) + B - B(0, y),$$
$$u = u(0, y) - \frac{1}{M_B{}^2}[B - B(0, y)]. \tag{2.1}$$

Here $1/M_B^2 = 1/M^2 + A^2 = c_B^2/v_0^2$ is the square of the ratio between the velocity of the fast magnetoacoustic wave propagating transverse to the magnetic field and the flow velocity.

System (2.1) has an obvious meaning: the flow of an ideally conducting gas transverse to a magnetic field is described in this approximation by the same equation as the flow of a neutral gas, except that the Mach number M is replaced by M_B.

When $M_B > 1$, system (2.1) has simple analytic solutions. If the channel cross section away from the plane $x = 0$ changes in such a manner that the lower wall is described by $y = f_1(x)$, while the upper wall is described by the straight line $y = f_2(x) = 1$, and if the flow is uniform in this cross section, the solution is given by

$$v = \sum_{n=0}^{\infty}\chi(x - 2nk - ky) - \sum_{n=1}^{\infty}\chi(x - 2nk + ky), \tag{2.2}$$

$$B = 1 + \frac{M_B{}^2}{k}\Big[\sum_{n=0}^{\infty}\chi(x - 2nk - ky) + \sum_{n=1}^{\infty}\chi(x - 2nk + ky)\Big]. \tag{2.3}$$

Here we have

$$\chi(x) = \begin{cases} f_1'(x) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \qquad (k = \sqrt{M_B{}^2 - 1}).$$

Figure 2 shows the electric-current lines, that is, the lines B = const (solid lines) and the gas streamlines (dashed lines) in the channel for $M_B^2 = 2$, $f_1 = -0.02x^2$, and $f_2 = 1$. In this case, the electric-current lines are straight lines with breaks, which are slight discontinuities. The current density normal to the channel wall changes abruptly at points separated by a distance $2(M_B^2 - 1)^{1/2}$.

When the flow parameters are averaged over a cross section, the usual linear-approximation relations for a quasi-one-dimensional flow are obtained. In particular, we find the following for the longitudinal velocity:

$$\langle u\rangle' = \frac{(f_2 - f_1)'}{M_B{}^2 - 1}, \qquad \langle u\rangle = \frac{1}{f_2 - f_1}\int_{f_1}^{f_2} u\,dy.$$

Here $f_2 - f_1$ is the distance between channel walls.

3. Solution (2.3) for the magnetic field does not satisfy boundary conditions (1.5) and (1.6). In particular, the current distribution in the flux core does not depend on whether the channel wall is a conductor or a nonconductor. The current redistribution required to satisfy the boundary conditions on the magnetic field should occur in the current boundary layer.

Under the assumption that this layer is thin, and with the standard (in boundary layer theory [6]) evaluations of the terms in Eqs. (1.3), one can simplify these equations. The induction equation for the current layer becomes

$$\varkappa^2\frac{\partial^2 B}{\partial y^2} = \frac{\partial B}{\partial x} - \frac{\partial B^\circ}{\partial x}. \tag{3.1}$$

Here and below, the superscript ° denotes quantities evaluated at the boundary between the flow layer and core. This equation can be rewritten as

$$\varkappa^2\frac{\partial^2 j_x}{\partial y^2} = \frac{\partial j_x}{\partial x}. \tag{3.2}$$

Solutions are given below for the layer at the lower channel wall, at which $y = 0$. The boundary condition on this layer at $y = 0$ is condition (1.5) for an insulator or (1.6) for an electrode; also, as $y \to \infty$, the solution should become equal to solution (2.3) for the lower boundary of the flow core. In addition, the initial conditions $B(0, y)$ and $j_x(0, y)$ must be specified to solve parabolic equations (3.1) and (3.2).

Accordingly, current-layer problems under these conditions are mathematically equivalent to problems involving linear heat flow [7] and have analogous solutions. On the other hand, this problem is very similar

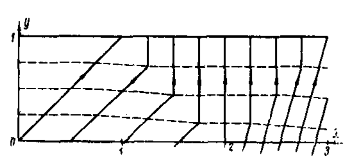

Fig. 2

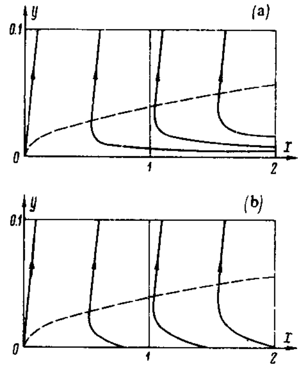

Fig. 3

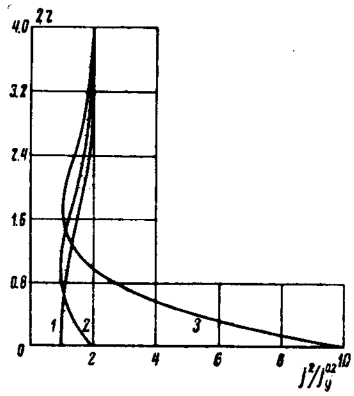

Fig. 4

to that of a viscous boundary layer with slipping, which was solved in the linear approximation in [8].

The field and current distributions near a nonconducting wall are given by Eq. (3.1) with the boundary conditions

$$B = 1 \quad \text{for } y = 0, \qquad B \to B^\circ \quad \text{for } y \to \infty,$$
$$B = B(0, y) \quad \text{for } x = 0.$$

In a channel of variable cross section, and in which the magnetic fields are described by (2.3), the solution in a layer at a nonconducting wall of parabolic form,

$$f_1 = {}^1/_2\, ax^2,$$

is, for $0 < x < 2(M_B^2 - 1)^{1/2}$

$$B = 1 + \frac{M_B^2 ax}{k}\Big\{1 - [1 - \Phi(r)](1 + 2r^2) - \frac{2}{\sqrt{\pi}}\, r \exp(-2r^2)\Big\}.$$

Here $\Phi(r)$ is the probability integral, and $r = y/2\varkappa\sqrt{x}$.

Figure 3a shows the electric-current lines in the layer at a nonconducting wall for $M_B^2 = 2$, $f_1 = -0.02x^2$, and $\varkappa = 10^{-2}$. In the layer, the current slips along the surface, and the current density is much greater than in the flow core. This leads to a greater rate of Joule energy dissipation in the current layer. The electromagnetic forces are perpendicular to the wall. In an expanding channel, in which a flow with $M_B > 1$ is accelerated, the current slipping along a nonconducting wall in the layer tends to detach the flow from the wall.

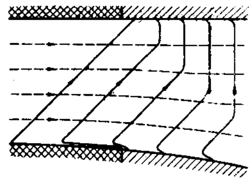

Fig. 5

The currents in layers at electrodes are described by Eq. (3.2) with boundary conditions

$$j_x = H(\varkappa^2\, \partial j_x/\partial y + \partial B^\circ/\partial x) \quad \text{for } y = 0,$$
$$j_x \to j_x^0 = -M_B^2 f'' \text{ for } y \to \infty,$$
$$j_x = j_x(0, y) = \psi(y) \quad \text{for } x = 0.$$

If the current is distributed in the flow core according to (2.3), and the electrode is of parabolic shape, $f_1 = ax^2/2$, the longitudinal current in the layer in the region $0 < x < 2(M_B^2 - 1)^{1/2}$ will be given by

$$j_x = aM_B^2 \frac{H}{\sqrt{M_B^2 - 1}} - aM_B^2\Big(\frac{H}{\sqrt{M_B^2 - 1}} + 1\Big)\{\Phi(r) + \exp(by + b^2\varkappa^2 x) \times$$
$$\times [1 - \Phi(r + b\varkappa\sqrt{x})]\} + \frac{1}{2\varkappa\sqrt{\pi x}}\int_0^\infty \Big[\exp\frac{-(y-\eta)^2}{4\varkappa^2 x} - \exp\frac{-(y+\eta)^2}{4\varkappa^2 x}\Big]\psi(\eta)\, d\eta,$$
$$b = \frac{1}{H\varkappa^2}.$$

The electric-current lines at an ideally conducting anode are shown in Fig. 3d for $M_B^2 = 2$, $f_1 = -0.02x^2$, $\varkappa = 10^{-2}$, $H = 30$, and $\psi = 0$.

The thickness of the current layer in both cases is of the order of

$$\delta = \frac{1}{\varkappa} = \frac{1}{\sqrt{R_m(1 + A^2M^2)}} = \frac{M_B}{M} R_m^{-1/2},$$

and differs by the factor $M_B/M = c_T/c_B \le 1$ from the layer thickness in an incompressible fluid which is a good conductor [1].

The Joule dissipation in the layer is more intense than in the core when $H > (M_B^2 - 1)1/2$. Near the electrode this dissipation increases in proportion to $(1 + H^2)$. Figure 4 shows the distribution of Joule dissipation in a layer at an anode for $(M_B^2 = 2$ [in this figure, 1) H = 0; 2) H = 1; 3) H = 3].

The variation in the electric current near a slight-discontinuity line (Fig. 2) can be described by an equation analogous to (3.2) and written in terms of coordinates including the slight-discontinuity line. This equation describes the structure of the slight discontinuity in a nonviscous, electrically conducting, and non-heat-conducting plasma; the direction of the electric-current lines changes continuously in a layer on the order of δ in thickness.

On the basis of this discussion, we can draw a picture of the currents and other flow parameters at the entrance to a channel formed by insulators and electrodes. Figure 5 shows the electric-current lines (solid curves) and the gas streamlines (dashed curves) for an expanding channel and for $M_B^2 = 2$, $f_1 = -0.04x^2$, $f_2 = 1$, $\varkappa = 10^{-2}$, and $H = 3$.

In conclusion, the author thanks A. I. Morozov for useful discussion of these results.

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